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An almost-Markovian Galilean-invariant turbulence model

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A model equation of Langevin type for the turbulent velocity field is constructed, in which the non-linear terms of the Navier–Stokes equation are replaced by a dynamical damping term and a random forcing term, with strength parameters determined by the past history of the energy spectrum. The model leads to a closed set of first-order differential equations in time for the evolution of two functions: the energy spectrum and the effective memory times for the interaction of mode triads. Invariance of the energy transfer to random Galilean transformation is achieved by using the interaction between solenoidal and compressive parts of a convected test field to determine the memory–time functions. The model equation is developed from the direct-interaction approximation as starting-point. At an intermediate stage, before the Galilean invariance is introduced, a model representation of Edwards's (1964) theory is obtained which extends the latter to statistically non-stationary states.

1. Motivation

Leith and Kraichnan have recently given a formulation of the direct-interaction approximation that features a model equation, for the velocity field, involving both a dynamical damping term, with memory, and a forcing term which depends on a purely random field (Kraichnan 1970). The latter field has the same covariance function as the velocity field. The conservative energy transfer in the turbulence then appears in the statistical equations as a balance between the damping and forcing terms. This model amplitude equation shows immediately the realizability properties of the direct-interaction covariance function, and it is a logical starting-point for using the direct-interaction approximation for estimating the effects of subgrid scales on larger scales in computer simulations of turbulence. Moreover, the solutions of the Navier–Stokes equation can be expanded about those of the model equation, thereby giving a power series from which systematic, converging corrections to the direct-interaction approximation can be constructed (Kraichnan 1970).

The work to be reported here started with an effort to modify the model amplitude equation to incorporate invariance to random Galilean transformations, along the lines of the Lagrangian-history direct-interaction approximation (Kraichnan 1965), and thereby give Kolmogorov's $k^{-\frac{5}{3}}$ inertial-range spectrum instead of a $k^{-\frac{3}{2}}$ spectrum. The only successful kind of modification which

emerged was much more drastic than originally intended; it involved making the random forcing term a white noise in time, and, correspondingly, reducing the memory of the damping term to zero. At an intermediate stage, before the feature of Galilean invariance is incorporated, the resulting Langevin equation gives a model representation of Edwards's (1964) 'time-independent' turbulence theory. The model amplitude equation at this stage is more properly called almost-Markovian, rather than Markovian, because the damping and forcing terms contain strength parameters that depend on past statistics of the velocity field.

The direct-interaction approximation fails to give a $k^{-\frac{5}{3}}$ inertial-range spectrum, because it yields effective dynamical times, limiting the build up of triple correlations, that are the order of the convective dephasing times associated with advection by the energy-containing scales. In the Lagrangian-history direct-interaction approximation, this deficiency was corrected by replacing the convective dephasing times by correlation times of the Lagrangian velocity field. In order to be able to compute the Lagrangian correlations, it was necessary to expand the direct-interaction statistical equations to a substantially larger and more complicated set, involving both Eulerian and Lagrangian covariances.

In the present work, a simpler way of incorporating random Galilean invariance was sought, in order to make the final equations more practicable for computing flows with more complicated geometry than isotropic turbulence, and for computing systematic corrections to the basic approximation. In the procedure finally adopted, the build-up times for energy-transferring triple correlations are measured by the times for dynamical interaction of the solenoidal and longitudinal parts of a random vector field convected by the turbulent velocity field. This permits a particularly simple set of closed equations in a purely Eulerian framework, and, as will be discussed later, does not seem any more arbitrary than the Lagrangian-history modification, once the necessity of an almost-Markovian model equation is assumed.

2. Summary of the direct-interaction equations

Let us write the forced Navier–Stokes equation for an infinite, incompressible fluid in the Fourier-space form,

$$(\partial/\partial t + \nu k^2)\, u_i(\mathbf{k}, t) = -\tfrac{1}{2} i P_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p}, t)\, u_m(\mathbf{q}, t) + f_i(\mathbf{k}, t). \tag{2.1}$$

Here the wave-vectors take all allowed values in a large cyclic box, and

$$P_{ijm}(\mathbf{k}) = k_m P_{ij}(\mathbf{k}) + k_j P_{im}(\mathbf{k}), \quad P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2.$$

We include the solenoidal stirring force f_i in order to permit statistically stationary isotropic turbulence. Except where stated otherwise, we restrict $f_i(\mathbf{k}, t)$ to white noise in time and to isotropy in space:

$$(L/2\pi)^3 \langle f_i(\mathbf{k}, t) f_j^*(\mathbf{k}, t') \rangle = 2 P_{ij}(\mathbf{k})\, Z(k, t)\, \delta(t - t'), \tag{2.2}$$

where $\langle\ \rangle$ denotes ensemble average. The direct-interaction model amplitude equation (Kraichnan 1970) is

$$(\partial/\partial t + \nu k^2)\, u_i(\mathbf{k}, t) + \int_0^t \eta(k, t, s)\, u_i(\mathbf{k}, s)\, ds = q_i(\mathbf{k}, t) + f_i(\mathbf{k}, t), \tag{2.3}$$

where

$$q_i(\mathbf{k}, t) = -iP_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \xi_j(\mathbf{p}, t)\, \xi'_m(\mathbf{q}, t), \tag{2.4}$$

$$\langle \xi_i(\mathbf{k}, t)\, \xi_j^*(\mathbf{k}, t')\rangle = \langle \xi'_i(\mathbf{k}, t)\, \xi_j'^*(\mathbf{k}, t')\rangle = \langle u_i(\mathbf{k}, t)\, u_j^*(\mathbf{k}, t')\rangle, \tag{2.5}$$

$$\eta(k, t, s) = \pi k \iint_\Delta b_{kpq} G(p, t, s)\, U(q, t, s)\, pq\, dp\, dq, \tag{2.6}$$

$$b_{kpq} = (p/k)\,(xy + z^3).$$

Here the random fields ξ_i and ξ'_i are statistically independent of each other and of the initial velocity field $u_i(\mathbf{k}, t = 0)$. $U(k, t, s)$ is the covariance scalar defined by

$$(L/2\pi)^3 \langle u_i(\mathbf{k}, t)\, u_j^*(\mathbf{k}, s)\rangle = \tfrac{1}{2} P_{ij}(\mathbf{k})\, U(k, t, s), \tag{2.7}$$

and $G(k, t, s)$ is the average infinitesimal response scalar (Kraichnan 1964*b*). The integration $\iint_\Delta$ is over all of the p, q plane where k, p, q can form a triangle, and x, y, z are the cosines of the interior angles opposite k, p, and q. L is a side of the cyclic box, and we take $L \to \infty$.

Both the fictitious forcing term $q_i(\mathbf{k}, t)$ and the damping function $\eta(k, t, s)$ depend on the velocity field only through ensemble averages. In the case of an infinite ensemble, they are unaffected by the value of the velocity field in any typical realization, so that (2.3) is effectively a linear dynamical equation for $u_i(\mathbf{k}, t)$. It follows immediately that the Green's function scalar of (2.3) satisfies

$$\left.\begin{aligned} (\partial/\partial t + \nu k^2)\, G(k, t, t') + \int_{t'}^t \eta(k, t, s)\, G(k, s, t')\, ds = 0 \quad (t \geqslant t'), \\ G(k, t', t') = 1, \end{aligned}\right\} \tag{2.8}$$

and that (2.3) yields the energy-balance and time-displaced covariance equations

$$\begin{aligned} &(\partial/\partial t + 2\nu k^2)\, U(k, t, t) + 2\int_0^t \eta(k, t, s)\, U(k, s, t)\, ds \\ &\quad = 2Z(k, t) + 2\pi k \int_0^t ds \iint_\Delta a_{kpq} G(k, t, s)\, U(p, t, s)\, U(q, t, s)\, pq\, dp\, dq, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &(\partial/\partial t + \nu k^2)\, U(k, t, t') + \int_0^t \eta(k, t, s)\, U(k, s, t')\, ds \\ &\quad = \pi k \int_0^{t'} ds \iint_\Delta a_{kpq} G(k, t', s)\, U(p, t, s)\, U(q, t, s)\, pq\, dp\, dq \quad (t \geqslant t'). \end{aligned} \tag{2.10}$$

Here $a_{kpq} = \frac{1}{2}[b_{kpq} + b_{kqp}]$. Equations (2.8)–(2.10) are the direct-interaction statistical equations.

The kinetic energy per unit mass is

$$\int_0^\infty E(k, t)\, dk = 2\pi \int_0^\infty U(k, t, t)\, k^2\, dk.$$

Equation (2.9) gives conservation of ensemble-averaged energy by the nonlinear interaction. However, energy is not conserved individually in the separate flow realizations that make up the ensemble, since (2.3) portrays the interaction as a steady dynamical damping competing with a fluctuating fictitious driving term.

The failure of the direct-interaction statistical equations (2.8)–(2.10) to keep the invariance properties of the Navier–Stokes equation under random Galilean transformations (the latter defined as a spatially uniform convection of the velocity field, varying randomly from realization to realization) has been analysed in detail (Kraichnan 1964*a*). We wish now to examine how this failing shows up in the model amplitude equation (2.3). A random Galilean transformation applied to (2.1) affects the phases of the Fourier amplitudes, and consequently makes smaller the fall-off times of the Eulerian averages $U(k,t,t')$ and $G(k,t,t')$ as functions of $t-t'$. But simultaneous triple correlations among interacting triads of wave-numbers are unaffected by the random convection, so that energy transfer is invariant. Random Galilean transformation on (2.3) induces qualitatively similar changes in the difference-time dependence of $U(k,t,t')$ and $G(k,t,t')$, but simultaneous triple correlations, and energy transfer, are not invariant. Two related properties are responsible for this. First, the quantities $\xi_i(\mathbf{k},t)$ and $\xi_i'(\mathbf{k},t)$ are strictly random variables, statistically independent for different $\mathbf{k}$ values. In distinction from $u_i(\mathbf{k},t)$, these functions cannot carry the higher statistical correlations associated with random coherent convection of the turbulent velocity field. Secondly, the induced change in $G(k,t,s)$ and $U(q,t,s)$, as functions of $t-s$, affects the contribution to $\eta(k,t,s)$ from each interacting triad k,p,q which enters (2.6). This, in turn, affects the energy transfer associated with each triad.

3. An almost-Markovian model amplitude equation

The Lagrangian-history direct-interaction energy balance equation differs from (2.9) in that the integrals back in time over Eulerian time-displaced covariances are replaced by integrals back along the trajectories of fluid elements. In this way, spurious effects of convective dephasing on the energy transfer are eliminated. The procedure is a hybridization of the original direct-interaction approximation; Lagrangian quantities are introduced *a posteriori* into an Eulerian statistical equation. The author, and several other workers, have searched unsuccessfully for a corresponding alteration of (2.3), which would provide a model representation of the Lagrangian-history direct-interaction equations. The attempts appear to fail because, in contrast to a covariance, in which just one of the factors averaged over can be changed from Eulerian to Lagrangian, an amplitude equation must be for either one field or the other. The almost-Markovian Eulerian model amplitude equation, to be described now, restores the freedom needed for alterations that remove convective dephasing effects from the energy transfer. This is done at the expense of ending with a qualitatively less faithful representation of time-displaced covariances.

Instead of (2.3), consider the model equation,

$$[\partial/\partial t+\nu k^2+\eta(k,t)]\,u_i(\mathbf{k},t) = q_i(\mathbf{k},t)+f_i(\mathbf{k},t), \tag{3.1}$$

where
$$\eta(k,t) = \pi k \int\int_{\Delta} b_{kpq}\theta_{pqk}(t)\, U(q,t)\, pq\, dp\, dq, \tag{3.2}$$

$$q_i(k,t) = -iP_{ijm}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} [\theta_{kpq}(t)]^{\frac{1}{2}}\, w(t)\, \xi_j(\mathbf{p},t)\, \xi'_m(\mathbf{q},t). \tag{3.3}$$

Here $w(t)$ is a white-noise process,

$$\langle w(t)\, w(t')\rangle = 2\delta(t-t'), \tag{3.4}$$

$U(q,t) \equiv U(q,t,t)$, and we continue to impose (2.5). The quantity $\theta_{kpq}(t)$ is a so-far undetermined characteristic memory time for the energetic interaction of wave-numbers k, p and q. Its presence takes the place of the explicit integration over history which occurs in (2.3).

Equation (3.1) gives the Green's function, energy, and time-displaced covariance equations,

$$[\partial/\partial t + \nu k^2 + \eta(k,t)]\, G(k,t,t') = 0 \quad (t \geqslant t'), \tag{3.5}$$

$$[\partial/\partial t + \nu k^2 + \eta(k,t)]\, U(k,t,t') = 0 \quad (t \geqslant t'), \tag{3.6}$$

$$[\partial/\partial t + 2\nu k^2 + 2\eta(k,t)]\, U(k,t) = 2Z(k,t) + 2\pi k \int\int_{\Delta} a_{kpq}\theta_{kpq}(t)\, U(p,t)\, U(q,t)\, pq\, dp\, dq. \tag{3.7}$$

From (3.5) and (3.6), we have

$$U(k,t,t') = G(k,t,t')\, U(k,t') \quad (t \geqslant t'). \tag{3.8}$$

In a statistically steady state, the direct-interaction energy equation (2.9) takes precisely the form (3.7) with

$$\theta_{kpq}(t) = \int_0^t G(k,t,s)\, r(p,t,s)\, r(q,t,s)\, ds, \quad r(k,t,s) \equiv U(k,t,s)/[U(k,t)\, U(k,s)]^{\frac{1}{2}} \tag{3.9}$$

(Kraichnan 1964). In view of (3.8), we therefore reproduce the *form* of the steady-state direct-interaction energy transfer if we complete the present equations with

$$\theta_{kpq}(t) = \int_0^t G(k,t,s)\, G(p,t,s)\, G(q,t,s)\, ds. \tag{3.10}$$

This does not mean, of course, that we get the same *values* for $G(k,t,s)$ as in the steady-state direct-interaction equations.

Equation (3.5) implies $G(k,t,t') \geqslant 0$, so that (3.10) ensures $\theta_{kpq}(t) \geqslant 0$. This is a necessary condition for (3.3) to make sense. We may note that any choice of $\theta_{kpq}(t)$ gives conservation of energy, while any choice symmetric in all three indices gives equipartition in absolute statistical equilibrium. Equation (3.8), which holds in general for the present model, is true for the direct-interaction approximation (and for the exact dynamics as well) only in absolute equilibrium. In a dissipative steady state (ν and Z not zero), the direct-interaction equations give $G(k,t,t') \neq r(k,t,t')$.

The direct interaction approximation gives a $U(k,t)$ whose Taylor expansion in powers of t agrees with that of the exact $U(k,t)$ through the term in t^3, provided that all triple correlations vanish at $t = 0$. This is true also for the almost-Markovian model if (3.10) is taken. If an ordering parameter is placed before the non-linear term in the Navier–Stokes equation (that is, $P_{ijm}(k)$ replaced by

$\lambda P_{ijm}(k))$, $U(k,t,t')$ is an even function of λ provided that all odd initial correlations vanish. The direct-interaction approximation gives $U(k,t,t')$ correctly through the term in λ^2, if U is expanded in powers of λ (essentially a Reynolds number expansion). The almost-Markovian model with (3.10) has the much more restricted property of giving $U(k,t)$ correctly through the term in λ^2 in a statistically steady state supported by a forcing field f_i that is white noise in time. The restriction on f_i (which we have assumed in the preceding) makes (3.8) true for the zeroth-order term in the λ expansion of the exact U.

A general deficiency of the almost-Markovian model is that it gives a qualitatively incorrect behaviour of $U(k,t,t')$ and $G(k,t,t')$, as a function of $t-t'$, for small $t-t'$, unless $t' = 0$. This appears to be an unavoidable consequence of replacing (2.3) by a Langevin equation, and shows up most clearly in the statistically steady state of absolute equilibrium (ν and Z zero, truncation in $\mathbf{k}$ space to keep energy contained). Then (3.5) and (3.6) give $G(k,t,t')$ and $U(k,t,t')$ a finite slope at $t = t'$, while the exact functions exhibit zero slope.

Differentiation of (3.10) and use of (3.5) gives

$$d\theta_{kpq}(t)/dt = 1 - [\nu(k^2+p^2+q^2) + \eta(k,t) + \eta(p,t) + \eta(q,t)]\,\theta_{kpq}(t), \qquad (3.11)$$

and (3.10) gives $\theta_{kpq}(0) = 0$. Thus, in contrast to the irreducible integro-differential equations of the direct-interaction approximation, the present model leads to a closed set of ordinary first-order differential equations in time, involving only $U(k,t)$, $\eta(k,t)$ and $\theta_{kpq}(t)$. These equations are (3.2), (3.7) and (3.11).

In a statistically steady state, $\eta(k,t) = \eta(k)$ and (3.11) gives

$$\theta_{kpq} = 1/[\nu(k^2+p^2+q^2) + \eta(k) + \eta(p) + \eta(q)]. \qquad (3.12)$$

The result is identical with Edwards's (1964) 'time-independent' theory. Thus, the present formulation may be regarded as extending Edwards's theory to non-steady states, and providing it with a model representation.

The model defined by (3.1)–(3.4) and (3.10) is still not invariant to random Galilean transformation. Since $q_i(\mathbf{k},t)$ is now a white noise in time, there can be no direct effect of random uniform convection on the frequency spectrum, or correlation time, of $q_i(\mathbf{k},t)$. However, the qualitative effects on the difference-time dependence of $G(k,t,t')$ and $U(k,t,t')$ are similar to those for the direct-interaction model. Thus, $\theta_{kpq}(t)$, and the contribution of each triad interaction to $\eta(k,t,s)$, are still affected, and, consequently, so is the energy transfer.

4. Galilean-invariant modification

An almost-Markovian analogue of the Lagrangian-history direct-interaction approximation can be constructed by first setting up the direct-interaction equations for the 'generalized velocity field' $u_i(\mathbf{k},t|t')$ (Kraichnan 1965). Then changes can be made similar to those which give (3.1)–(3.3), and, finally, $\theta_{kpq}(t)$ can be taken in a form like (3.10), but with Lagrangian rather than Eulerian Green's functions in the integral. The freedom of choice of θ_{kpq} in the almost-Markovian models allows many ways to do this, in fact, whereas, in the original Lagrangian-history theory, the modifications of the direct-interaction equations

were highly constrained by invariance properties. This lack of uniqueness suggests that the simplest possible way of eliminating spurious convection effects should be sought.

The model to be described now originates with the observation that advection does not directly decorrelate the Lagrangian velocity, since the particle acceleration is wholly a response to pressure, viscous force and external stirring. How, then, can the decorrelating effect of pressure forces be expressed in a purely Eulerian formulation? That is, how can we isolate, from advection effects, the distortion of the Eulerian velocity field by pressure? We cannot simply throw out the advection term, since then there would be no pressure forces generated. Instead, we can reverse the question and ask what the pressure forces prevent; what would happen if pressure were switched off. Non-uniform advection in the absence of pressure would, of course, distort an initially solenoidal velocity field so that it developed a longitudinal part, or compressive component. With pressure present, the kinetic energy that otherwise would be converted into the compressive field appears instead as a distortion in shape and direction of the solenoidal field. This suggests that we take, as a measure of the distorting effect of pressure, the rate at which advection would induce interplay between solenoidal and compressive components with pressure absent.

The simplest formal expression of the proposal just stated appears to be the following. Let $v_i(\mathbf{k}, t)$ be a vector field subject to viscous damping and advected by a solenoidal velocity field $\hat{u}_i(\mathbf{k}, t)$ according to

$$(\partial/\partial t + \nu k^2)\, v_i(\mathbf{k}, t) = -i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} k_m \hat{u}_m(\mathbf{q}, t)\, v_i(\mathbf{p}, t), \tag{4.1}$$

where we use $q_m \hat{u}_m(\mathbf{q}, t) = 0$. The solenoidal and compressive parts of v_i are

$$v_i^S(\mathbf{k}, t) = P_{ij}(\mathbf{k})\, v_j(\mathbf{k}, t), \quad v_i^C(\mathbf{k}, t) = \Pi_{ij}(\mathbf{k})\, v_j(\mathbf{k}, t), \tag{4.2}$$

where $\Pi_{ij} = k_i k_j / k^2$. We now retain on the right-hand side of (4.1) only that part which represents interaction of $\mathbf{v}^S$ and $\mathbf{v}^C$; that is, we throw out the terms coupling $\mathbf{v}^S$ and $\mathbf{v}^C$ to themselves. The result is

$$(\partial/\partial t + \nu k^2)\, v_i^S(\mathbf{k}, t) = -ik_m \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} P_{ij}(\mathbf{k})\, v_j^C(\mathbf{p}, t)\, \hat{u}_m(\mathbf{q}, t), \tag{4.3}$$

$$(\partial/\partial t + \nu k^2)\, v_i^C(\mathbf{k}, t) = -ik_m \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \Pi_{ij}(\mathbf{k})\, v_j^S(\mathbf{p}, t)\, \hat{u}_m(\mathbf{q}, t). \tag{4.4}$$

The interaction terms in (4.3) and (4.4) conserve

$$\sum_{\mathbf{k}} [|\mathbf{v}^S(\mathbf{k}, t)|^2 + |\mathbf{v}^C(\mathbf{k}, t)|^2].$$

An almost-Markovian model equation can now be constructed from (4.3) and (4.4), in the same way that the model of § 3 is constructed from the Navier–Stokes equation. The average Green's-function tensors for $\mathbf{v}^S$ and $\mathbf{v}^C$ have the form,

$$\left.\begin{aligned} G_{ij}^S(\mathbf{k}, t, t') = P_{ij}(\mathbf{k})\, G^S(k, t, t'), \quad G_{ij}^C(\mathbf{k}, t, t') = \Pi_{ij}(\mathbf{k})\, G^C(k, t, t'), \\ G^S(k, t', t') = 1, \quad G^C(k, t', t') = 1. \end{aligned}\right\} \tag{4.5}$$

The model equations for G^S and G^C are

$$\left.\begin{aligned}[\partial/\partial t+\nu k^2+\eta^S(k,t)]\,G^S(k,t,t') &= 0,\\ [\partial/\partial t+\nu k^2+\eta^C(k,t)]\,G^C(k,t,t') &= 0,\end{aligned}\right\} \tag{4.6}$$

where

$$\left.\begin{aligned}\eta^S(k,t) &= \pi k\iint_\Delta b^G_{kpq}\theta^G_{kqp}(t)\,U(q,t)\,pq\,dp\,dq,\\ \eta^C(k,t) &= 2\pi k\iint_\Delta b^G_{kpq}\theta^G_{kqp}(t)\,U(q,t)\,pq\,dp\,dq,\end{aligned}\right\} \tag{4.7}$$

and

$$\theta^G_{kpq}(t) = \int_0^t G^C(k,t,s)\,G^S(q,t,s)\,G^S(p,t,s)\,ds, \tag{4.8}$$

with

$$b^G_{kpq} = \tfrac{1}{2}(1-y^2)\,(1-z^2). \tag{4.9}$$

In (4.7), we have taken $\hat{U}(q,t) = U(q,t)$ so as to make the modal intensities of the so far unspecified $\hat{\mathbf{u}}$ field the same as those of the Navier–Stokes field $\mathbf{u}$. In (4.8), we take

$$\hat{G}(q,t,s) = G^S(q,t,s)$$

so as to tie the decorrelation behaviour of the $\hat{\mathbf{u}}$ field to that of the field $\mathbf{v}^S$. This hybrid assignment of properties to the $\hat{\mathbf{u}}$ field takes the place of the hybridization of statistical equations inherent in the Lagrangian-history direct-interaction procedure. To complete the set of equations, we finally replace (3.10) by

$$\theta_{kpq}(t) = \int_0^t G^S(k,t,s)\,G^S(p,t,s)\,G^S(q,t,s)\,ds, \tag{4.10}$$

thereby assuming that the G^S function may be taken as a measure of the distortions that limit the build up of energy-transferring triple correlations.

The geometrical coefficient b^G_{kpq} arises, of course, in the direct-interaction equations for G^S and G^C, and is carried to the almost-Markovian equations, as in § 3. The factor of 2 in the equation (4.7) for $\eta^C(k,t)$ arises because each compressive degree of freedom interacts with two transverse degrees of freedom for any wave-vector. In two-dimensional isotropic turbulence, there is only one transverse degree of freedom, and the corresponding results are

$$G^S(k,t,t') = G^C(k,t,t'),$$

with

$$\eta^S(k,t) = \eta^C(k,t) = 4k^2\iint_\Delta (1-x^2)^{-\frac{1}{2}}\,b^G_{kpq}\theta^G_{kpq}(t)\,U(q,t)\,dp\,dq, \tag{4.11}$$

where b^G_{kpq} is the same and kinetic energy per unit mass is

$$\int_0^\infty \pi U(q,t)\,q\,dq.$$

By differentiating (4.8) and (4.10), we obtain

$$d\theta^G_{kpq}(t)/dt = 1-[\nu(k^2+p^2+q^2)+\eta^C(k,t)+\eta^S(p,t)+\eta^S(q,t)]\,\theta^G_{kpq}(t), \tag{4.12}$$

$$d\theta_{kpq}(t)/dt = 1-[\nu(k^2+p^2+q^2)+\eta^S(k,t)+\eta^S(p,t)+\eta^S(q,t)]\,\theta_{kpq}(t). \tag{4.13}$$

The final closed set of equations is then (3.2), (3.7), (4.7), (4.12) and (4.13), with $\theta^G_{kpq}(0) = \theta_{kpq}(0) = 0$.

The preceding equations were obtained by a succession of inductive steps, and it would be pointless to try to defend the derivation strongly. In common with many ideas associated with inertial dynamics, the physical concepts to which we appealed resist precise formulation. However, it is easy to demonstrate the self-consistency of the final results. The energy equation is unaltered in form from § 3, so that conservation of energy is not lost; moreover, the symmetry of $\theta_{kpq}(t)$ in its indices assures equipartition in absolute statistical equilibrium. Equation (4.6) shows that G^S and G^C are always positive, so that $\theta_{kpq}(t)$ and $\theta^G_{kpq}(t)$ are always positive. Also, (4.9) gives $b^G_{kpq} \geqslant 0$ (in distinction to b_{kpq}), and it follows that $G^S(k,t,s)$ and $G^C(k,t,s)$ are monotonically decreasing functions of t.

Invariance of the energy transfer in the final equations to random Galilean transformation follows directly from the fact that addition of a uniform velocity field to $\hat{\mathbf{u}}$ makes no contribution to the right-hand sides of (4.3) and (4.4). In the final equations, the invariance shows up in the fact that, for $q \ll k$, we have $b^G_{kpq} \sim \frac{1}{2}q^2k^{-2}(1-y^2)$, so that the contribution of small q to (4.7) is proportional to the mean-square shear in the low wave-numbers, rather than, as in (3.2), to the kinetic energy.

The present equations reproduce the expansion of the exact $U(k,t)$ in powers of t and in powers of the ordering parameter λ through the same orders as does the model of § 3.

The model amplitude equation defined by (3.1)–(3.4) still belongs to the present set of statistical equations ((3.2), (3.7) (4.7), (4.12), (4.13)) in the sense that it yields (3.7). Although convective dephasing effects of large scales have been removed from the energy transfer, $\eta(k,t)$ is still the order of the reciprocal of the convective dephasing time for wave-number k, and the model $u_i(\mathbf{k},t)$ still displays characteristic fluctuation times appropriate to the Eulerian velocity field. These facts are consistent, because the a_{kpq} and b_{kpq} terms in (3.7) give a cancellation of the contributions of the energy-containing range to the energy budget at high k.

5. Rescaling of memory times

The direct-interaction equations are fully determined from the Navier–Stokes equation, so that there is no place for adjustable parameters. This is not true, however, for the Galilean-invariant almost-Markovian model developed in § 4. Assuming that the physical reasoning is valid, the characteristic times obtained from (4.3) and (4.4) are at best a *measure* of the build-up times for triple correlations in the Navier–Stokes system. The model is equally plausible if we scale the characteristic times by putting a constant factor g, of order one, in the right-hand sides of (4.3) and (4.4). The value of g could be fixed by seeking a best fit of the model to some aptly chosen computer experiment or laboratory data. However, we can also fix g, wholly within the framework of derivation of the model, by requiring a fit to the original direct-interaction results in a case where the latter can reasonably be expected to be good.

Since the worst failing of the direct-interaction approximation appears to be the representation of the interaction of widely different scales, we choose, for this purpose, the dynamical system obtained by eliminating from the Navier–

Stokes equation all wave-vectors that do not lie in a thin spherical shell of radius k. Then all the modes which remain are dynamically identical. For the interactions within the shell, we have $b_{kpq} \approx b_{kkk} = \frac{3}{8}$, $b^G_{kpq} \approx b^G_{kkk} = \frac{9}{32} = \frac{3}{4}b_{kkk}$. Consider the state of absolute equilibrium (ν and Z zero), in which (3.8) holds for the direct-interaction functions and (3.10) is the form of θ_{kpq} in the direct-interaction energy balance equation. Suppose now that energy is very slowly fed into some of the modes in the shell by weak external forces. Then the rate at which this energy is distributed to the rest of the modes is determined by the absolute-equilibrium value of θ_{kkk}.

In the equilibrium state, all the quantities in (2.6) depend only on time difference, and this equation reduces to the form,

$$\eta(t-s) = Kb_{kkk}[G(t-s)]^2,$$

where $\eta(t-s) = \eta(k,t,s)$ and $G(t-s) = G(k,t,s)$. The parameter K depends on the thickness of the shell and the mode intensity $U(k,t)$. For convenience, let us choose the latter so that $K = \frac{8}{3}$ and (2.8) reduces to

$$dG(t)/dt = -\int_0^t [G(t-s)]^2 G(s)\, ds \quad (G(0) = 1). \tag{5.1}$$

Equation (5.1) is easily solved numerically. For small t the solution resembles $e^{-t^2/2}$, but it goes slightly negative for $2.5 < t < 5.5$, then becomes positive again, and eventually dies in exponential fashion. The numerical solution gives for $\theta_{kkk}(\infty)$ the value

$$\theta_{DI} = 0{\cdot}72335.$$

The scaling factor g inserted in (4.3) and (4.4) puts a factor g^2 in the right-hand sides of (4.7) and (4.11), and otherwise leaves (4.5) to (4.14) unchanged. With the same normalization that gives (5.1), we find that (4.7), (4.12) and (4.13) reduce to

$$\eta^S = \tfrac{3}{4}g^2\theta^G, \quad \eta^C = \tfrac{3}{2}g^2\theta^G, \quad \theta^G = 1/(2\eta^S + \eta^C), \quad \theta = 1/(3\eta^S). \tag{5.2}$$

The solution is

$$\eta^S = \tfrac{1}{4}3^{\frac{1}{2}}g, \quad \eta^C = 2\eta^S, \quad \theta^G = 1/(3^{\frac{1}{2}}g), \quad \theta = \tfrac{4}{3}\theta^G, \tag{5.3}$$

so that the value of θ matches the direct-interaction value if

$$g = \tfrac{4}{3}(\theta_{DI}\, 3^{\frac{1}{2}})^{-1} = 1.064. \tag{5.4}$$

This choice of g then assures that the model of § 4 gives the same results as the direct-interaction approximation for the energy flow associated with slowly varying, small deviations from absolute equilibrium in the spherical shell. It is interesting, but apparently not significant, that the value of g is so close to one. It *is* significant for the plausibility of the model, however, that this same value of g gives a value for Kolmogorov's constant that agrees fairly well with experiment. The inertial-range applications of the model are discussed in an accompanying paper.

6. Discussion

The almost-Markovian model of § 4 offers the advantages, over the direct-interaction approximation, of incorporating invariance of energy transfer to random Galilean transformation and of being much simpler to integrate. It has the advantages, over the Lagrangian-history direct-interaction approximation, of simplicity and representation by a model amplitude equation. As noted already, it has these virtues at the expense of giving a qualitatively unfaithful representation of time-displaced covariances for small time differences. The almost Markovian model is perhaps best regarded as a tool for following the evolution of spectral intensities only, with the other functions occurring in the equations considered as intermediate quantities.

Even if only energy transfer is considered, the almost-Markovian models of both §§ 3 and 4 display differences in behaviour from the direct-interaction approximation in the portrayal of transient effects, and these differences are of a kind which cannot be removed by a simple rescaling like that carried out in § 5. In general, current energy transfer according to the direct-interaction equations is more influenced by the past history of energy disequilibrium than according to the almost-Markovian models. An indication of this is the damping-with-memory that appears explicitly in (2.3). As a consequence, strong initial disequilibrium tends to produce overshoot effects in the direct-interaction energy transfer which are absent in the almost-Markovian models. It is unclear to what extent this is good or bad, from the point of view of reproducing exact dynamical behaviour. On the other hand, if a sudden transient change in the spectrum $U(k, t)$ is produced (by external forces) at a time $t_1 > 0$, the almost-Markovian models give a clearly artificial initial response of the energy transfer. In both the direct-interaction and exact dynamics, the initial change in energy transfer is $\propto (t - t_1)$ if the perturbation velocity field is initially uncorrelated with the existing field. In the almost-Markovian models, there is a discontinuity in the transfer function, because transfer of both the existing and perturbation spectrum is indiscriminately controlled by the functions $\theta_{kpq}(t_1)$. Nevertheless, the model of § 4 appears to give qualitatively correct decay times for the spectrum transient, while the direct-interaction approximation and the model of § 3 give qualitatively wrong decay times at high wave-numbers because of the lack of Galilean invariance.

The saving in machine computation time offered by the almost-Markovian models is very impressive. In a direct-interaction computation of spectrum evolution, most of the computation effort is in computing time-displaced functions, and the computer time increases as the cube of the time of evolution, unless integrals over history are truncated. In the model of § 4, most of the computation time is taken up by computing the energy equation, with (4.12) and (4.13) taking comparable but somewhat less time because they are algebraically simpler than the energy equation. The total computation time is proportional to time of evolution. Computation time for the almost-Markovian models is, in fact, only slightly greater than for the single-time quasi-normal approximation (Proudman & Reid 1954).

The split of a vector field into solenoidal and longitudinal parts is possible in

any geometry, once the boundary conditions are fixed. A consequence is that the model of § 4 can be generalized to inhomogeneous flows with non-zero mean fields. In this respect, it offers strong advantages over the simpler procedure of fixing θ_{kpq} in isotropic turbulence by dimensional arguments, whose generalization to other cases is then unclear. The model can also be extended in a nice way to handle spectral transport of a passive scalar. We carry out the direct-interaction approximation for the problem and construct an almost-Markovian model in analogy to § 3. Then we make a correspondence between the compressive part $\mathbf{v}^C$ of the test field and the gradient field of the scalar. Thus, the very same $G^C(k, t, s)$ as in § 4 is used to measure the distortion of the scalar field. This procedure is free of a deficiency of the Lagrangian-history direct-interaction treatment of scalar transport, that was noted when the latter was first formulated (Kraichnan 1965, p. 596). In the absence of molecular diffusivity, scalar density is constant along the particle trajectories, so that Lagrangian scalar correlations are unlikely to provide a satisfactory measure of the distortions that limit transport. The almost-Markovian model may correct the over-estimates of scalar transport efficiency that have been found in computations of the Lagrangian-history direct-interaction approximation (Kraichnan 1968).

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